Duality and Pomeron effective theory for QCD at high energy and large N_c

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Abstract

We propose an effective theory which governs Pomeron dynamics in QCD at high energy, in the leading logarithmic approximation, and in the limit where N_c, the number of colors, is large. In spite of its remarkably simple structure, this effective theory generates precisely the evolution equations for scattering amplitudes that have been recently deduced from a more complete microscopic analysis. It accounts for the BFKL evolution of the Pomerons together with their interactions: dissociation (one Pomeron splitting into two) and recombination (two Pomerons merging into one). It is constructed by exploiting a duality principle relating the evolutions in the target and the projectile, more precisely, splitting and merging processes, or fluctuations in the dilute regime and saturation effects in the dense regime. The simplest Pomeron loop calculated with the effective theory is free of both ultraviolet or infrared singularities.

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There has been recently significant progress in our understanding of high energy hadronic scattering, and in particular of the processes occurring at large parton densities and which are believed to be responsible for the unitarization of the scattering amplitudes and the saturation of the parton distributions. Non linear evolution equations have been derived which describe the approach towards saturation and the unitarity limit, and which have the structure of stochastic evolution equations. However, it has been very recently recognized [1] that the equations which were considered as the most complete, namely the Balitsky–JIMWLK (Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner) equations [2–5], are in fact incomplete. This is manifest in the statistical language by the presence of fluctuations at high momenta [6, 7] which are not well accounted for by the JIMWLK evolution of the target wavefunction [1]. In the language of Pomerons, the JIMWLK equation contains Pomeron merging but not also Pomeron splitting [2].

Following this observation, two of us (E.I. and D.T.) have constructed a hierarchy of non–linear evolution equations for the dipole scattering amplitudes which include both gluon mergings and gluon splittings, and thus generate Pomeron loops through iterations [1, 8]. These equations have been argued to hold in the limit where the number of colors N_c is large ($N_c \gg 1$), and indeed it has been checked explicitly in Ref. [8] that the vertices appearing in these equations are the same as the corresponding 'triple Pomeron vertices' computed in perturbative QCD at large N_c [9–11]. A complementary approach has been developed by Mueller, Shoshi, and Wong [12] who proposed a generalization of the JIMWLK equation which includes the effects of pomeron splitting in the dilute regime and for large N_c. These two approaches follow the same general strategy — namely, they combine the non–linear JIMWLK equation at high density with the color dipole picture [13, 14] in the dilute regime — and lead indeed to the same evolution equations for the scattering amplitudes, as demonstrated in Ref. [8]. (See also Refs. [15, 16] for related recent developments.)

It is our purpose in this letter to show that the equations obtained in [1, 8] and [12] can be reformulated in term of an effective theory for Pomerons. By 'Pomeron' we mean here the color singlet exchange which describes the interaction between an elementary color dipole and the field of a target in a single scattering approximation, and which reduces to two gluon exchanges in lowest order perturbation theory. The construction of the effective theory involves a projection onto restricted degrees of freedom, precisely the Pomerons, and is expressed in terms of a simple Hamiltonian which describes the BFKL evolution of the Pomerons together with their splitting and merging. By requiring that the evolution should lead to identical results whether it is viewed as the evolution of the target or that of the projectile, one arrives at a duality principle which is used to construct the effective Hamiltonian from the Hamiltonian derived in [12] in the dilute regime. The limitations of the effective theory, and the subtle mathematical problems that arises when one attempts to analyze its microscopic content will be briefly discussed at the end of this letter.

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[2] Note that the opposite terminology for what one means by 'splitting' and 'merging' would be more natural in relation with Balitsky equations, which refer to the evolution of the projectile. To avoid confusion on this point, in this paper we shall systematically use the terminology appropriate to target evolution.

Most treatments of high energy scattering rely on an asymmetric approach: typically, the 'projectile' is viewed as a collection of test particles which probe the color field of the 'target'. At high energy, the eikonal approximation is a good approximation, and the scattering of an elementary color charge is described by a *'Wilson line'* of the form

$$V_{\boldsymbol{x}}^{\dagger}[\alpha] \equiv \mathrm{P} \exp\left(ig \int dx^- \alpha^a(x^-, \boldsymbol{x}) t^a\right), \tag{1}$$

where $\boldsymbol{x}$ denotes the transverse coordinate of the particle, which is not affected by its interactions with the field of the target $\alpha^a(x^-, \boldsymbol{x})$, t^a are the generators of the SU(3) algebra in the representation appropriate for the test particle, and the symbol P indicates that, in the expansion of the exponential, the color matrices $\alpha^a(x^-, \boldsymbol{x}) t^a$ must be ordered from right to left in increasing order in x^- (we are using light–cone vector notations, $x^{\pm} \equiv (t \pm z)/\sqrt{2}$.). For a more complex projectile, viewed as a collection of elementary color charges, the S–matrix is given by a product of Wilson lines like Eq. (1), one for each elementary color charge.

In a frame in which most of the total rapidity Y is carried by the target, the target wavefunction can be described as a *color glass condensate* [4, 17], and the corresponding S–matrix is obtained as:

$$\langle S \rangle_Y = \int \mathrm{D}[\alpha]\, W_Y[\alpha]\, S[\alpha], \tag{2}$$

where $\alpha \equiv \alpha^a(x^-, \boldsymbol{x})$ is a classical field randomly distributed with *weight function $W[\alpha]$* (a functional probability distribution), and $S[\alpha]$ is the projectile S–matrix for a given configuration of this random field. With increasing Y, the weight function evolves according to a functional renormalization group equation, of the generic form:

$$\frac{\partial}{\partial Y} W_Y[\alpha] = -H\left[\alpha, \frac{\delta}{i\delta\alpha}\right] W_Y[\alpha], \tag{3}$$

where H is a functional differential operator commonly referred to as the 'Hamiltonian'. Alternatively, one can view the same evolution as a change in the scattering operator, for a fixed weight function $W[\alpha]$. To see that, take a derivative w.r.t. Y in Eq. (2), use Eq. (3), and perform an integration by part in the functional integral:

$$\frac{\partial}{\partial Y} \langle S \rangle_Y = -\int \mathrm{D}[\alpha]\, W[\alpha]\, H^{\dagger}\left[\alpha, \frac{\delta}{i\delta\alpha}\right] S[\alpha]. \tag{4}$$

This can be interpreted as describing the evolution of the scattering operator $S_Y[\alpha]$, with 'Hamiltonian' $H^{\dagger}$:

$$\frac{\partial}{\partial Y} S_Y[\alpha] = -H^{\dagger}\left[\alpha, \frac{\delta}{i\delta\alpha}\right] S_Y[\alpha]. \tag{5}$$

Both points of view, somewhat reminiscent of, respectively, the Schrödinger and the Heisenberg pictures of quantum mechanics, will be used in the following discussion (although we shall refrain from introducing explicitly rapidity dependent operators). In the Schrödinger picture, one puts emphasis on the evolution of the state vector, whose role is played here by the weight functional $W_Y[\alpha]$. In the Heisenberg picture, the state vector is a constant reference vector involved in the calculation of all expectation values, here $W[\alpha]$, and one puts all the evolution in the operators, here the scattering operators $S_Y[\alpha]$. The Schrödinger picture corresponds to evolution equations which aim at providing a detailed microscopic description of the color field in the target, together with its complicated correlations. This is what the JIMWLK equation does. The Heisenberg picture rather describes how the test particles get dressed by color field fluctuations as they are boosted to higher rapidities. In this approach, the complicated color correlations in the target wavefunction are not immediately visible, and indeed the resulting equation of motion are established somewhat more easily. This second approach is essentially the one used by Balitsky to obtain his hierarchy of equations.

The test particles that we shall consider are in fact elementary color dipoles, whose scattering amplitude reads:

$$T(\boldsymbol{x}, \boldsymbol{y}) = 1 - \frac{1}{N_c} \mathrm{tr}(V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}}), \tag{6}$$

for a dipole with the quark leg at $\boldsymbol{x}$ and the antiquark leg at $\boldsymbol{y}$. Here the Wilson lines are taken in the fundamental representation. We shall be interested in situations where the dipoles scatter off the color glass in the two-gluon exchange approximation (weak field limit) and we shall work in a large-N_c limit. In the weak field limit, the amplitude for a single dipole to scatter is obtained after expanding each of the Wilson lines to second order in α:

$$\begin{aligned}
V_{\boldsymbol{x}}^\dagger[\alpha] =\ & 1 + ig \int dx^- \alpha^a(x^-, \boldsymbol{x}) t^a \\
& - \frac{g^2}{2} \int dx^- \int dy^- \alpha^a(x^-, \boldsymbol{x}) \alpha^b(y^-, \boldsymbol{x}) \Big[\theta(x^- - y^-) t^a t^b + \theta(y^- - x^-) t^b t^a \Big] \\
& + \cdots .
\end{aligned} \tag{7}$$

Note that, to this order, the x^-–ordering of the color matrices starts to play a role in Eq. (7). Still, this ordering is irrelevant for the computation of the dipole amplitude to lowest order, because of the symmetry of the color trace: $\mathrm{tr}(t^a t^b) = \frac{1}{2} \delta^{ab} = \mathrm{tr}(t^b t^a)$. Namely, one finds:

$$T(\boldsymbol{x}, \boldsymbol{y}) \simeq T_0(\boldsymbol{x}, \boldsymbol{y}) \equiv \frac{g^2}{4N_c} \left[\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y}) \right]^2, \tag{8}$$

which involves only the integrated field $\alpha^a(\boldsymbol{x}) \equiv \int dx^- \alpha^a(x^-, \boldsymbol{x})$. Similarly the amplitude for κ dipoles to scatter is given, within the same approximation, by $T_0^{(\kappa)}(\boldsymbol{x}_1, \boldsymbol{y}_1, ..., \boldsymbol{x}_\kappa, \boldsymbol{y}_\kappa) =$

$T_0(\boldsymbol{x}_1, \boldsymbol{y}_1)...T_0(\boldsymbol{x}_\kappa, \boldsymbol{y}_\kappa)$. In what follows, we shall refer to the amplitude (8) describing the single–scattering of an elementary dipole off a given color field as to a "Pomeron exchange". Similarly, $T_0^{(\kappa)}$ describes the exchange of κ Pomerons.

At this point we find it useful to digress on the linear evolution equation known as BFKL equation. This will allow a few observations which illuminate some of the mathematical subtleties involved in taking the large N_c limit when constructing our effective theory. Consider first the JIMWLK Hamiltonian. As shown in Ref. [18], when it is restricted to act on gauge-invariant observables, it can be given the simple form:

$$H_{\mathrm{JIMWLK}} = -\frac{1}{16\pi^3} \int\limits_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \left(1 + \widetilde{V}_{\boldsymbol{u}}^\dagger \widetilde{V}_{\boldsymbol{v}} - \widetilde{V}_{\boldsymbol{u}}^\dagger \widetilde{V}_{\boldsymbol{z}} - \widetilde{V}_{\boldsymbol{z}}^\dagger \widetilde{V}_{\boldsymbol{v}} \right)^{ab}$$
$$\times \frac{\delta}{i\delta\alpha_Y^a(\boldsymbol{u})} \frac{\delta}{i\delta\alpha_Y^b(\boldsymbol{v})}, \tag{9}$$

where $\mathcal{M}$ is the dipole kernel

$$\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}. \tag{10}$$

Here the Wilson lines are in the adjoint representation. The derivatives can be freely moved across the bilinear form in Wilson lines, because they commute with the latter in the presence of the dipole kernel. That is, H_{JIMWLK} is Hermitian. The BFKL limit of H_{JIMWLK} is obtained by expanding the Wilson lines to lowest non-trivial order in α. One gets:

$$H_{\mathrm{BFKL}} = -\frac{g^2}{16\pi^3} \int\limits_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})[\alpha^a(\boldsymbol{u}) - \alpha^a(\boldsymbol{z})][\alpha^b(\boldsymbol{v}) - \alpha^b(\boldsymbol{z})]$$
$$\times f^{acf} f^{bfd} \frac{\delta}{\delta\alpha_Y^c(\boldsymbol{u})} \frac{\delta}{\delta\alpha_Y^d(\boldsymbol{v})}. \tag{11}$$

It is not difficult to verify that H_{BFKL} is again Hermitian.

Let us now turn to the 'large–N_c limit'. This is obtained by (i) restricting the action of H_{BFKL} to the dipole operators $T_0^{(\kappa)}$ mentioned above and (ii) preserving only the dominant terms at large N_c in the action of the Hamiltonian on these operators. When acting on the color fields inside a single factor T_0 (i.e., on the same dipole), the two functional derivatives in H_{BFKL} yield a factor δ^{cd}, and then $f^{acf} f^{bfc} = -N_c\delta^{ab}$ produces the expected N_c enhancement. On the other hand, the action on the color fields within two different factors T_0 (i.e., upon two different dipoles) produces no such enhancement. Thus, at large N_c, H_{BFKL} can be equivalently replaced by an effective Hamiltonian in which the two functional derivatives are traced over color. This Hamiltonian, which we denote $H_0^\dagger$ for reason which will become clear shortly, is

$$H_0^\dagger = \frac{1}{2N_c^2} \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \left[\alpha^a(\boldsymbol{u}) - \alpha^a(\boldsymbol{z})\right] \left[\alpha^a(\boldsymbol{v}) - \alpha^a(\boldsymbol{z})\right] \frac{\delta}{\delta\alpha^b(\boldsymbol{u})} \frac{\delta}{\delta\alpha^b(\boldsymbol{v})}, \tag{12}$$

where $\bar{\alpha}_s = \alpha_s N_c/\pi$. Let us emphasize that, as obvious from the construction we have given, the two derivatives in $H_0^\dagger$ are to act on *the same dipole*. Note also that, as opposed to the original H_{BFKL}, $H_0^\dagger$ is not Hermitian: in fact, it is readily seen that its adjoint is ill-defined. This reflects the fact that the construction of $H_0^\dagger$ involves a projection on a specific set of degrees of freedom, and once this is done, one looses the possibility to integrate by part as in Eq. (4) in order to let H_0 act on the weight functional $W[\alpha]$. These special mathematical properties, restriction of the space on which the Hamiltonian is acting and loss of hermiticity, are general, and peculiar, mathematical features of the effective theory that we shall present. It is tempting to speculate that in doing the large N_c limit we are renouncing to follow the evolution of some color correlations (precisely those which are suppressed at large N_c). The corresponding loss of information may be responsible for the simpler Markovian stochastic theory that we shall arrive at.

We now return to the main stream of our discussion and establish a useful property. In Ref. [19], a symmetric description was obtained for the scattering between two color glasses in the regime where *both* systems are in the weak field regime). The final formula reads

$$\langle S \rangle_Y = \int \mathrm{D}[\alpha_R] \, W_{Y-y}[\alpha_R] \int \mathrm{D}[\alpha_L] \, W_y[\alpha_L] \, \mathrm{e}^{i \int d^2 \boldsymbol{z} \, \rho_L^a(\boldsymbol{z}) \alpha_R^a(\boldsymbol{z})} \, . \tag{13}$$

In this expression, $\rho_L^a(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}^2 \alpha_L^a(\boldsymbol{x})$ is the classical color charge density of the left–mover, and $W_{Y-y}[\alpha_R]$ and $W_y[\alpha_L]$ are the weight functions for the right–moving and, respectively, left–moving color glass (note that the rapidity of the left mover is measured positively to the left, so that as we vary y, the total rapidity interval between projectile and target remains equal to Y). The precise conditions for the validity of Eq. (13) are detailed in Ref. [19]. Let us emphasize here a non-trivial aspect of this formula. Although it is essentially a weak field formula which assumes that the elementary dipoles interact only once, it contains the possibility that any number of dipoles of the projectile interact with an equivalent number of dipoles in the target. Thus Eq. (13) does account for multiple scattering, albeit in a restrictive way (each dipole interacting only once). These multiple scattering generate unitarity corrections if Y is large enough. At the same time, we require both color glasses to be unsaturated. This imposes some limited range of variation for y within which Eq. (13) is correct.

Now, Lorentz invariance implies that $\langle S_Y \rangle$ may depend on the total rapidity interval Y, but, within the range of validity of Eq. (13), cannot depend upon the rapidity y used to separate the system into a 'projectile' and a 'target', or equivalently on the frame which we choose to describe the collision. This implies (see also Ref. [20] for a similar argument):

$$0 = \frac{\partial \langle S \rangle_Y}{\partial y} = \int \mathrm{D}[\alpha_R] \int \mathrm{D}[\alpha_L] \, \mathrm{e}^{i \int d^2 \boldsymbol{z} \, \rho_L^a(\boldsymbol{z}) \alpha_R^a(\boldsymbol{z})}$$

$$\left\{ \left(\frac{\partial}{\partial y} W_{Y-y}[\alpha_R] \right) W_y[\alpha_L] + W_{Y-y}[\alpha_R] \left(\frac{\partial}{\partial y} W_y[\alpha_L] \right) \right\} \, . \tag{14}$$

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The evolution of both weight functions are given by:

$$\frac{\partial}{\partial y} W_{Y-y}[\alpha_R] = -\frac{\partial}{\partial Y} W_{Y-y}[\alpha_R] = H\left[\alpha_R, \frac{\delta}{i\delta\alpha_R}\right] W_{Y-y}[\alpha_R],$$

$$\frac{\partial}{\partial y} W_y[\alpha_L] = -H\left[\alpha_L, \frac{\delta}{i\delta\alpha_L}\right] W_y[\alpha_L]. \tag{15}$$

We shall keep the evolution of the left–mover as shown in the above equation, but perform an integration by parts in the functional integral over α_R in Eq. (14). Next, we note that

$$H^\dagger\left[\alpha_R, \frac{\delta}{i\delta\alpha_R}\right] e^{i\int d^2\mathbf{z}\,\rho_L^a(\mathbf{z})\alpha_R^a(\mathbf{z})} = H\left[\frac{\delta}{i\delta\rho_L}, \rho_L\right] e^{i\int d^2\mathbf{z}\,\rho_L^a(\mathbf{z})\alpha_R^a(\mathbf{z})}. \tag{16}$$

Using this identity in Eq. (14) and performing a further integration by parts, now w.r.t. α_L (recall that $\rho_L^a(\mathbf{x}) = -\nabla_{\mathbf{x}}^2\alpha_L^a(\mathbf{x})$), one is left with a differential operator acting on $W_y[\alpha_L] \equiv W_y[\rho_L]$ (with a slight abuse in the notation):

$$H^\dagger\left[\frac{\delta}{i\delta\rho_L}, \rho_L\right] W_y[\rho_L]. \tag{17}$$

For Eq. (14) to be satisfied, the contribution above should cancel against the term in Eq. (15) describing the evolution of the left–mover. This condition leads to the 'self-duality' condition:

$$H\left[\alpha_L, \frac{\delta}{i\delta\alpha_L}\right] W_y[\alpha_L] = H^\dagger\left[\frac{\delta}{i\delta\rho_L}, \rho_L\right] W_y[\rho_L]. \tag{18}$$

The same relation holds obviously for the 'right' variables α_R, ρ_R.

Going back to Eq. (16), one sees that what is involved in the duality operation[3] is a matching of splitting processes in the left movers, encoded by terms in the Hamiltonian of the form $\rho^2\delta^n/\delta\rho^n$, into merging process in the right movers, corresponding to terms of the form $\alpha^n\delta^2/\delta\alpha^2$. An example of such a matching is illustrated in Fig. 1. Splitting terms dominate in the dilute regime where they control the fluctuations, while merging terms become essential in the saturation regime where parton densities are large. This *fluctuation–saturation duality* is turned into a constraint on the evolution Hamiltonian of either the projectile or the target in Eq. (18).

The self-duality constraint, which we expect to hold within the limited range of energies in which the factorization (13) is valid[4] [19], will be used now to construct a simple

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[3] To our knowledge, the duality between the roles of the operators $\rho^2\delta^n/\delta\rho^n$ and $\alpha^n\delta^2/\delta\alpha^2$ has been first recognized by L. McLerran.

[4] The self-duality condition (15) has recently been claimed to hold in a much broader context [20].
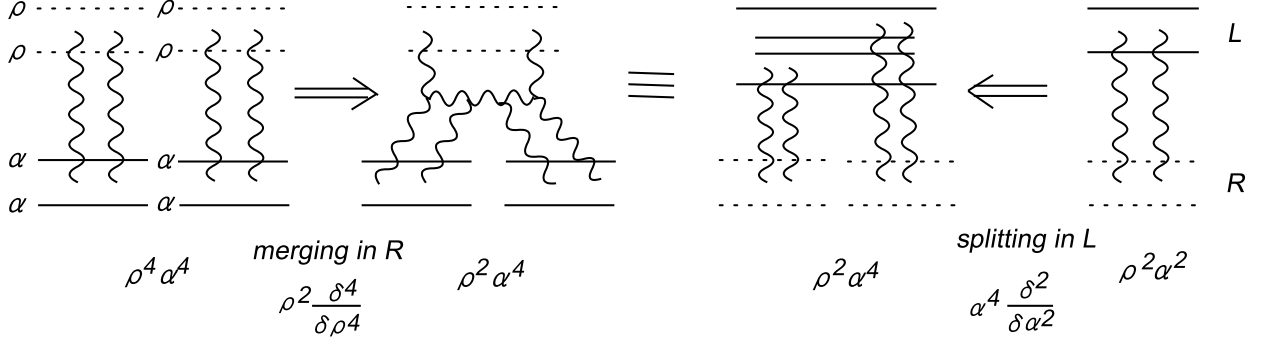
Fig. 1. An illustration of the relation (16). ρ denotes the color charge of the left–mover (L) and α is the color field of the right–mover (R). The same diagram (any of the two diagrams in the middle) can be viewed either as a merging in R, or as a splitting in L. The first interpretation is natural when the diagram is produced by acting on the eikonal line with the Hamiltonian $H^\dagger_{1\to2} \sim \rho^2\delta^4/\delta\rho^4$ for L (cf. Eq. (19)). The second interpretation rather corresponds to the action of $H^\dagger_{2\to1} \sim \alpha^4\delta^2/\delta\alpha^2$ for R (cf. Eq. (24).

Hamiltonian describing the Pomeron dynamics in the dilute regime, starting from the known, dominant, contribution containing only splitting processes that has been constructed by Mueller, Shoshi and Wong [12]. Quite remarkably, and somewhat unexpectedly, this Hamiltonian leads to equations of motion which reproduces the exact ones at large N_c [1,8], that is the effective theory appears to be valid beyond the dilute limit where it is established. The Hamiltonian constructed in [12] reads

$$H_{1\to2} = -\frac{g^2}{16N_c^3}\frac{\bar{\alpha}_s}{2\pi}\int \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\mathcal{G}(\boldsymbol{u}_1|\boldsymbol{u},\boldsymbol{z})\mathcal{G}(\boldsymbol{v}_1|\boldsymbol{u},\boldsymbol{z})\mathcal{G}(\boldsymbol{u}_2|\boldsymbol{z},\boldsymbol{v})\mathcal{G}(\boldsymbol{v}_2|\boldsymbol{z},\boldsymbol{v})$$

$$\times \frac{\delta}{\delta\alpha^a(\boldsymbol{u}_1)}\frac{\delta}{\delta\alpha^a(\boldsymbol{v}_1)}\frac{\delta}{\delta\alpha^b(\boldsymbol{u}_2)}\frac{\delta}{\delta\alpha^b(\boldsymbol{v}_2)}\nabla_{\boldsymbol{u}}^2\nabla_{\boldsymbol{v}}^2\alpha^c(\boldsymbol{u})\alpha^c(\boldsymbol{v}). \qquad (19)$$

In Eq. (19), the integration goes over all the transverse coordinates $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{z}$, $\boldsymbol{u}_1$, $\boldsymbol{v}_1$, $\boldsymbol{u}_2$, $\boldsymbol{v}_2$. The function $\mathcal{G}(\boldsymbol{u}_1|\boldsymbol{u},\boldsymbol{z})$ is, up to a factor $g\,t^a$, the classical field created at $\boldsymbol{u}_1$ by the elementary dipole $(\boldsymbol{u},\boldsymbol{z})$, and reads

$$\mathcal{G}(\boldsymbol{u}_1|\boldsymbol{u},\boldsymbol{z}) = \frac{1}{4\pi}\ln\frac{(\boldsymbol{u}_1-\boldsymbol{z})^2}{(\boldsymbol{u}_1-\boldsymbol{u})^2}. \qquad (20)$$

It is easy to understand (and was explicitly shown in [8]) that this Hamiltonian generates *Pomeron splittings*. More precisely, the result of the operation of $H^\dagger_{1\to2}$ on the two–Pomeron exchange amplitude $T_0^{(2)}$ is proportional to T_0, and thus generates the following, *fluctuation*, term in the evolution equation for $T_0^{(2)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\boldsymbol{x}_2,\boldsymbol{y}_2)$:

$$H^\dagger_{1\to2}T_0^{(2)} = \left(\frac{\alpha_s}{2\pi}\right)^2\frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}}\mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\,\mathcal{A}_0(\boldsymbol{x}_1,\boldsymbol{y}_1|\boldsymbol{u},\boldsymbol{w})\,\mathcal{A}_0(\boldsymbol{x}_2,\boldsymbol{y}_2|\boldsymbol{w},\boldsymbol{v})\,\nabla_{\boldsymbol{u}}^2\nabla_{\boldsymbol{v}}^2\,T_0(\boldsymbol{u},\boldsymbol{v}),$$

$$\qquad (21)$$

where $\alpha_s^2\mathcal{A}_0$ is the amplitude for dipole–dipole scattering in the two–gluon exchange ap-

proximation and for large N_c :

$$\mathcal{A}_0(\boldsymbol{x}, \boldsymbol{y} | \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{8} \left[\ln \frac{(\boldsymbol{x} - \boldsymbol{v})^2 (\boldsymbol{y} - \boldsymbol{u})^2}{(\boldsymbol{x} - \boldsymbol{u})^2 (\boldsymbol{y} - \boldsymbol{v})^2} \right]^2. \tag{22}$$

Clearly, this process corresponds to the splitting of one Pomeron into two. In general the Hamiltonian $H_{1\to2}$ can describe the transition $n \to n + 1$, in which case $n - 1$ of the Pomerons are simply "spectators". Note that $H_{1\to2}$ is non-Hermitian, which we interpret as reflecting again the large–N_c approximation implicitly involved in its derivation.

To apply the duality transformation, it is convenient to reexpress $H_{1\to2}$ in terms of the sources $\rho^a(\boldsymbol{x})$ of the color field $\alpha^a(\boldsymbol{x})$, by using $\rho^a(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}^2 \alpha^a(\boldsymbol{x})$. We then obtain

$$H_{1\to2} = -\frac{g^2}{16N_c^3} \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}} \mathcal{M}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z})$$

$$\times \left[\frac{\delta}{\delta\rho^a(\boldsymbol{u})} - \frac{\delta}{\delta\rho^a(\boldsymbol{z})} \right]^2 \left[\frac{\delta}{\delta\rho^b(\boldsymbol{z})} - \frac{\delta}{\delta\rho^b(\boldsymbol{v})} \right]^2 \rho^c(\boldsymbol{u})\rho^c(\boldsymbol{v}). \tag{23}$$

At this point we force the Hamiltonian to be self-dual. This is done by adding to $H_{1\to2}[\delta/i\delta\rho, \rho]$ its dual $H_{1\to2}^\dagger[\alpha, \delta/i\delta\alpha] \equiv H_{2\to1}$ (this new notation will be justified shortly). The Hermitian conjugate of $H_{2\to1}$ reads

$$H_{2\to1}^\dagger = \frac{g^2}{16N_c^3} \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}} \mathcal{M}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) \left[\alpha^a(\boldsymbol{u}) - \alpha^a(\boldsymbol{z})\right]^2 \left[\alpha^b(\boldsymbol{z}) - \alpha^b(\boldsymbol{v})\right]^2 \frac{\delta}{\delta\alpha^c(\boldsymbol{u})} \frac{\delta}{\delta\alpha^c(\boldsymbol{v})}, \tag{24}$$

and the action of $H_{2\to1}^\dagger$ on the dipole scattering amplitude is

$$H_{2\to1}^\dagger T_0(\boldsymbol{x}, \boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \frac{g^4}{16N_c^2} [\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{z})]^2 [\alpha^b(\boldsymbol{z}) - \alpha^b(\boldsymbol{y})]^2$$

$$= \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) T_0^{(2)}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{z}, \boldsymbol{y}). \tag{25}$$

Thus $H_{2\to1}$ generates the non–linear term in the first Balitsky equation. Similarly, it is obvious to show that the operation on $T_0^{(\kappa)}(\boldsymbol{x}_1, \boldsymbol{y}_1; ...; \boldsymbol{x}_\kappa, \boldsymbol{y}_\kappa)$, will generate correctly the non-linear terms of the κ–the Balitsky equation in the large–N_c limit (this is trivial; only one amplitude is "active", and we need to take into account all the possible permutations). Therefore the Hamiltonian in Eq. (24) generates in an effective way *Pomeron mergings* (hence the notation $H_{2\to1}$); one has a transition of the form $n+1 \to n$ where, again, $n-1$ of the Pomerons are spectators.

Thus the total Hamiltonian of our Pomeron effective theory reads

$$H^\dagger = H_0^\dagger + H_{1\to2}^\dagger + H_{2\to1}^\dagger. \tag{26}$$

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The Hamiltonian $H_0^\dagger$, describing the BFKL evolution, plays here the role of the free Pomeron Hamiltonian. The other two pieces $H_{2\to 1}^\dagger$ and $H_{1\to 2}^\dagger$ correspond respectively to Pomeron merging and splitting, and will naturally generate *Pomeron loops* in the course of the evolution. The minimal Pomeron loop, which is simply the one–loop correction to the scattering amplitude $\langle T(\boldsymbol{x}, \boldsymbol{y})\rangle_Y$, can be isolated by the successive operation of these two parts of the Hamiltonian, namely $\mathbb{PL} = H_{1\to 2}^\dagger H_{2\to 1}^\dagger T_0$. The explicit results reads

$$\mathbb{PL} = -\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \int_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z},\boldsymbol{w}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \tag{27}$$
$$\times \mathcal{A}_0(\boldsymbol{x},\boldsymbol{z}|\boldsymbol{u},\boldsymbol{w})\mathcal{A}_0(\boldsymbol{z},\boldsymbol{y}|\boldsymbol{w},\boldsymbol{v})\nabla_{\boldsymbol{u}}^2\nabla_{\boldsymbol{v}}^2 \langle T_0(\boldsymbol{u},\boldsymbol{v})\rangle_Y .$$

Note that this result is free of any (ultraviolet or infrared) divergences. For instance, the poles in the dipole kernel at $\boldsymbol{z} = \boldsymbol{x}$ and $\boldsymbol{y}$ are harmless because of $\mathcal{A}_0(\boldsymbol{x},\boldsymbol{z} = \boldsymbol{x}|\boldsymbol{u},\boldsymbol{w}) = 0$.

A simple physical picture of this result is obtained by assuming that this Pomeron loop has been generated after the first two steps in the evolution starting with a target which is itself an elementary dipole $(\boldsymbol{x_0}, \boldsymbol{y_0})$. Then, Eq. (27) simplifies to:

$$\mathbb{PL}^0 = -2\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \alpha_s^4 \int_{\boldsymbol{z},\boldsymbol{w}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\mathcal{M}(\boldsymbol{x_0},\boldsymbol{y_0},\boldsymbol{w})\mathcal{A}_0(\boldsymbol{x},\boldsymbol{z}|\boldsymbol{x_0},\boldsymbol{w})\mathcal{A}_0(\boldsymbol{z},\boldsymbol{y}|\boldsymbol{w},\boldsymbol{y_0}). \tag{28}$$

This result has a clear physical interpretation: Both original dipoles — in the projectile and the target — split into new dipoles, processes which are represented by the two dipole kernels times $\bar{\alpha}_s^2$. Then, the child dipoles from the two systems scatter with each other, by exchanging two pairs of gluons; this yields the two factors $\mathcal{A}_0$ times α_s^4. Finally, note that this contribution is negative, as expected, leading to a decrease in the amplitude in the course of the evolution.

As we have already emphasized, the Hamiltonian (26) reproduces the complete equations of motion established in [1, 8] and [12]. While this intriguing property deserves further investigation, some insight can be gained by analyzing how the merging processes in the effective Hamiltonian compare to those deduced from correct microscopic dynamics as described by JIMWLK. The action of H_{JIMWLK}, Eq. (9), on the full dipole scattering amplitude $T(\boldsymbol{x}, \boldsymbol{y})$, Eq. (6), is

$$\frac{\delta}{\delta\alpha^a(\boldsymbol{u})}\frac{\delta}{\delta\alpha^b(\boldsymbol{v})}T(\boldsymbol{x},\boldsymbol{y}) = \frac{g^2}{N_c}(\delta_{\boldsymbol{yv}} - \delta_{\boldsymbol{xv}})\left[\delta_{\boldsymbol{ux}}\operatorname{tr}(t^b t^a V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}}) - \delta_{\boldsymbol{uy}}\operatorname{tr}(t^a t^b V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}})\right] \tag{29}$$

Simple algebra then easily yields the first Balitsky equation :

$$H_{\mathrm{JIMWLK}}T(\boldsymbol{x},\boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})[-T(\boldsymbol{x},\boldsymbol{y}) + T(\boldsymbol{x},\boldsymbol{z}) + T(\boldsymbol{z},\boldsymbol{y}) - T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})].$$
$$\tag{30}$$

Then, after expanding the dipole operator T in the weak-field limit, and keeping terms up to the quartic order with respect to gauge field α, one finds an evolution equation which

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contains not only the BFKL dynamics, but also the lowest order mergings (four gluons merging into two).

Consider now the action of the JIMWLK Hamiltonian on $T_0(\boldsymbol{x}, \boldsymbol{y})$. Since:

$$\frac{\delta}{\delta\alpha^a(\boldsymbol{u})}\frac{\delta}{\delta\alpha^b(\boldsymbol{v})}T_0(\boldsymbol{x}, \boldsymbol{y}) = \frac{g^2}{2N_c}\delta^{ab}(\delta_{\boldsymbol{xu}} - \delta_{\boldsymbol{yu}})(\delta_{\boldsymbol{xv}} - \delta_{\boldsymbol{yv}}), \tag{31}$$

we have:

$$H_{\mathrm{JIMWLK}}T_0(\boldsymbol{x}, \boldsymbol{y}) = \frac{g^2}{2N_c^2}\frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}}\mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\mathrm{Tr}\left(1 + \widetilde{V}_{\boldsymbol{x}}^\dagger\widetilde{V}_{\boldsymbol{y}} - \widetilde{V}_{\boldsymbol{x}}^\dagger\widetilde{V}_{\boldsymbol{z}} - \widetilde{V}_{\boldsymbol{z}}^\dagger\widetilde{V}_{\boldsymbol{y}}\right). \tag{32}$$

When expanding the Wilson lines in the r.h.s. in powers of α, one obtains quadratic terms describing the BFKL evolution of $T_0(\boldsymbol{x}, \boldsymbol{y})$ plus higher order terms which describe $n \to 2$ gluon mergings. But at this level, it is easy to see that the $4 \to 2$ terms generated by this expansion are not the same as those in the r.h.s. of Eq. (25). For instance, while the merging term in Eq. (25) includes a piece containing three different transverse positions (i.e., $\alpha_{\boldsymbol{x}}^a\alpha_{\boldsymbol{z}}^a\alpha_{\boldsymbol{z}}^b\alpha_{\boldsymbol{y}}^b$), the corresponding JIMWLK result in Eq. (32) cannot generate such terms.

We thus see that thet actual, microscopic, dynamics of gluon merging in QCD is considerably more complicated than in our simple effective theory, yet the latter provides, as we have seen, the correct evolution equations for the scattering amplitudes. This shows that the additional merging terms generated by the JIMWLK Hamiltonian must compensate in the evolution equations against non–linear (quartic in α, or higher) contributions to the scattering amplitudes, as obtained by expanding the Wilson lines in equations like (6). We can refer to the latter as describing the *dressing of the pomeron with multiple scattering*.

This brings us to comment on the nature of the dynamics described by the effective theory. This theory generates evolution equations for the Pomeron operators $T_0^{(\kappa)}$ which are formally identical to the equations satisfied by the complete dipole scattering operators $T^{(\kappa)}$ in QCD at large N_c. This means in particular that the solutions to the equations for $\langle T_0^{(\kappa)}\rangle_Y$ will appear to saturate the unitarity (or 'black disk') limit $T_0 = 1$ in the high energy limit, in spite of the fact that the respective operators describe single scatterings only ! This indicates that one must be extremely careful in the physical interpretation of the effective theory.

Let us then have a closer look at the *microscopic* dynamics that is describes. Effectively, the evolution of the target reduces to that of a system of *dipoles* subjected to a dynamics of a *reaction–diffusion* type: the dipoles undergo BFKL dynamics, they can split (one dipole into two dipoles), and they can also recombine with each other (two dipoles into one). The dynamics of such a system of dipoles is entirely coded in the k–body densities $n_Y^{(k)}$ (see Sect. 5 in Ref. [1] for a precise definition). Although we shall not work this out explicitly here, it is not hard, by using the results of Ref. [19] to relate these dipole densities to colorless correlation functions of the color charge density ρ^a. For instance the

dipole number operator $n(\boldsymbol{x}, \boldsymbol{y})$ can be identified with the bilocal operator $\rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})$ of the effective theory. With such identifications, and by using Eq. (26), it is straightforward to construct the evolution equations satisfied by the dipole densities. One thus finds that $n_Y(\boldsymbol{x}, \boldsymbol{y})$ obeys the BFKL equation supplemented by a negative term proportional to $n_Y^{(2)}$, which is generated by the merging piece $H_{2\to1}^\dagger$ of the Hamiltonian. Furthermore, the r.h.s. of the equation for $\partial n_Y^{(2)}/\partial Y$ includes the standard BFKL terms describing the individual evolutions of the two dipoles $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$, but also a positive, *fluctuation term*, proportional to n_Y — this is generated by the splitting piece $H_{1\to2}^\dagger$ of the Hamiltonian, and is the same as the corresponding term deduced from the dipole picture in Refs. [1, 8] — and, finally, a negative, *recombination*, term proportional to $n_Y^{(3)}$. We thus obtain an infinite hierarchy, which describes a dipole reaction–diffusion dynamics, as anticipated, and predicts the saturation of the dipole density at a value of order $1/\alpha_s^2$.

Now, it is clear that this is only an *effective dynamics* since, as well known, dipoles in real QCD do not simply recombine with each other: the interaction between two dipoles inside the target wavefunction goes beyond the large–N_c approximation and leads to more complicated color configurations, involving higher color multipoles [13, 14]. The reason why it has been possible to *simulate* the non–linear effects responsible for unitarity corrections in the equations for the scattering amplitudes through simple 'dipole recombination' processes in the target wavefunction is because the same non–linear effects can be interpreed as *projectile* evolution, in which case they describe the *splitting* of a dipole in the projectile. Then, the $1 \to 2$ dipole splitting vertex from the projectile is simply reinterpreted, within the effective theory, as a $2 \to 1$ 'dipole merging' vertex in the target. Note finally that a similar dipole model including splitting and recombination has been recently used in Ref. [15] to generate evolution equations with Pomeron loops. The present work shows how this *effective* dynamics may indeed emerge from the actual target dynamics in QCD, and points to numerous subtleties involved in this precise connection.

Acknowledgments

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